**Final Exam — Functional Analysis (WBMA033-05)**

Friday 4 April 2025, 11.45–13.45h

University of Groningen

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**Instructions**

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If $p$ is the number of marks then the exam grade is $G = 1 + p/10$.

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**Problem 1 (10 points)**

The linear space $\mathcal{C}([0,1], \mathbb{K})$ can be equipped with the following norms:

$$\|f\|_a = \int_0^1 |f(x)|\,dx \quad \text{and} \quad \|f\|_b = \int_0^1 x|f(x)|\,dx.$$

Are these norms equivalent? Motivate your answer.
Hint: consider the functions $f_n$ defined by $f_n(x) = 1 - nx$ for $x \in [0, 1/n]$ and $f_n(x) = 0$ for $x \in [1/n, 1]$.

**Problem 2 (10 + 10 + 10 = 30 points)**

Recall the following Banach space from the lecture notes:

$$\ell^\infty = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K}, \quad \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}, \quad \|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|.$$

Let $\alpha \in \mathbb{K}$ satisfy $|\alpha| < 1$ and consider the following linear operator:

$$T : \ell^\infty \to \ell^\infty, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \alpha^2 x_2, \alpha^3 x_3, \dots).$$

(a) Compute the operator norm of $T$.

(b) Prove that $T$ is compact by considering a suitable sequence $T_k \to T$.

(c) Prove that the spectrum of $T$ is given by $\sigma(T) = \{\alpha^n \,:\, n \in \mathbb{N}\} \cup \{0\}$.

**Problem 3 (10 points)**

Let $X$ be a linear space over $\mathbb{K} = \mathbb{C}$. Assume that $\langle \cdot, \cdot \rangle$ is an inner product on $X$ and denote the induced norm by $\|\cdot\|$. Show that for all $x, y \in X$ we have

$$4\langle x, y\rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2.$$

Hint: first compute $\lambda\|x + \lambda y\|^2$ for an arbitrary $\lambda \in \mathbb{C}$.

*Turn page for problems 4 and 5!*

**Problem 4 (5 + (5 + 5 + 5) = 20 points)**

(a) Formulate the Uniform Boundedness Principle.

(b) Consider the following normed linear space of all polynomials:

$$\mathcal{P} = \left\{ p(x) = \sum_{k=0}^{\infty} a_k x^k \ : \ a_k \in \mathbb{K} \text{ nonzero for only finitely many } k \right\},$$
$$\|p\| = \max_{k \geq 0} |a_k|.$$

For every $n \in \mathbb{N}$ consider the following linear map:

$$T_n : \mathcal{P} \to \mathbb{K}, \quad T_n p = \sum_{k=0}^{n} a_k.$$

Prove the following statements:

(i) For each $n \in \mathbb{N}$ we have $\|T_n\| = n + 1$.
Hint: consider the polynomial $p(x) = 1 + x + x^2 + \cdots + x^n$.

(ii) For each $p \in \mathcal{P}$ there exists a constant $C_p \geq 0$ such that $|T_n p| \leq C_p$ for all $n \in \mathbb{N}$;

(iii) The space $(\mathcal{P}, \|\cdot\|)$ is *not* a Banach space.

**Problem 5 (5 + (8 + 7) = 20 points)**

(a) Formulate the Hahn-Banach Theorem for normed linear spaces.

(b) Consider the space $\mathcal{C}([0,1], \mathbb{K})$ with the sup-norm. Fix $c \in [0,1]$ and consider the following linear maps:

$$f : \mathcal{C}([0,1], \mathbb{K}) \to \mathbb{K}, \qquad f(\varphi) = \int_0^1 \varphi(t)\, dt,$$

$$g : \mathcal{C}([0,1], \mathbb{K}) \to \mathbb{K}, \qquad g(\varphi) = \varphi(c).$$

(i) Show that $\|f\| = 1$ and $\|g\| = 1$.

(ii) Consider the linear subspace $V = \operatorname{span}\{1, x\}$ and the linear map

$$h : V \to \mathbb{K}, \qquad h(a + bx) = a + b/2.$$

Apply the Hahn-Banach Theorem to $h$: is the object of which the existence is asserted by that theorem unique?

**End of test (90 points)**

**Solution of problem 1 (10 points)**

Computing the $\|\cdot\|_a$-norm of $f_n$ gives

$$\|f_n\|_a = \int_0^1 |f_n(x)|\,dx = \int_0^{1/n} 1 - nx\,dx = \left[x - \frac{nx^2}{2}\right]_0^{1/n} = \frac{1}{2n}.$$

**(3 points)**

Computing the $\|\cdot\|_b$-norm of $f_n$ gives

$$\|f_n\|_b = \int_0^1 x|f_n(x)|\,dx = \int_0^{1/n} x - nx^2\,dx = \left[\frac{x^2}{2} - \frac{nx^3}{3}\right]_0^{1/n} = \frac{1}{6n^2}.$$

**(3 points)**

If the two norms are equivalent, then there exist constants $0 < m \leq M$ such that

$$m\|f\|_b \leq \|f\|_a \leq M\|f\|_b \quad \text{for all} \quad f \in \mathcal{C}([0,1],\mathbb{K}).$$

**(1 point)**

In particular, for the functions $f_n$ we obtain the inequality

$$\frac{1}{2n} \leq \frac{M}{6n^2} \quad \text{for all} \quad n \in \mathbb{N},$$

or, equivalently,

$$n \leq \frac{M}{3} \quad \text{for all} \quad n \in \mathbb{N}.$$

This would imply that the set of natural numbers is bounded which is clearly a contradiction. Therefore, there cannot exist a constant $M > 0$ such that $\|f\|_a \leq M\|f\|_b$ holds for all $f \in \mathcal{C}([0,1],\mathbb{K})$. We conclude that the two norms are not equivalent.
**(3 points)**

**Solution of problem 2 (10 + 10 + 10 = 30 points)**

(a) Since $|\alpha| < 1$ it follows that $|\alpha|^n \le |\alpha|$ for each $n \in \mathbb{N}$. Let $x \in \ell^\infty$ be arbitrary, then

$$\|Tx\|_\infty = \sup_{n\in\mathbb{N}} |\alpha^n x_n| = \sup_{n\in\mathbb{N}} |\alpha|^n \, |x_n| \le |\alpha| \sup_{n\in\mathbb{N}} |x_n| = |\alpha| \, \|x\|_\infty.$$

**(5 points)**

We conclude that

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \le |\alpha|.$$

**(3 points)**

Note that for $x = (1, 0, 0, \dots)$ we have $\|x\|_\infty = 1$ and $\|Tx\|_\infty = |\alpha|$ which implies that the operator norm of $T$ is given by $\|T\| = |\alpha|$.

**(2 points)**

(b) Define for $k \in \mathbb{N}$ the operator

$$T_k : \ell^\infty \to \ell^\infty, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \dots, \alpha^k x_k, 0, 0, 0, \dots)$$

The same argument as in part (a) shows that $T_k$ is bounded (and in fact we have $\|T_k\| = |\alpha|$ for all $k \in \mathbb{N}$). In addition, $\operatorname{ran} T_k$ is finite-dimensional. From Lemma 4.44 in the lecture notes it follows that $T_k$ is compact.

**(3 points)**

For any $k \in \mathbb{N}$ and $x \in \ell^\infty$ we have

$$\|(T - T_k)x\|_\infty = \sup_{n\geq k} |\alpha^n x_n| = \sup_{n\geq k} |\alpha^n| \, |x_n| \le |\alpha|^k \sup_{n\geq k} |x_n| \le |\alpha|^k \|x\|_\infty.$$

**(3 points)**

We conclude that $\|T - T_k\| \le |\alpha|^k$ for all $k \in \mathbb{N}$ and thus $T_k \to T$ in the space $B(\ell^\infty)$. Since each $T_k$ is compact it follows from Theorem 4.46 (or Corollary 4.47) in the lecture notes that $T$ is compact as well.

**(4 points)**

(c) Clearly, $\alpha^n$ is an eigenvalue of $T$ for each $n \in \mathbb{N}$. The corresponding eigenvector is given by the $n$-th standard unit vector. We conclude that $\{\alpha^n : n \in \mathbb{N}\} \subset \sigma(T)$.

**(2 points)**

Note that $\alpha^n \to 0$ since $|\alpha| < 1$. Since the spectrum is closed it follows that $0 \in \sigma(T)$.

**(1 point)**

If $\lambda \notin \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$, then there exists $\delta > 0$ such that $|\lambda - \alpha^n| \ge \delta$ for all $n \in \mathbb{N}$. Note that the inverse of $T - \lambda$ is given by

$$(T - \lambda)^{-1} x = \left( \frac{x_1}{\alpha - \lambda}, \frac{x_2}{\alpha^2 - \lambda}, \frac{x_3}{\alpha^3 - \lambda}, \dots \right).$$

**(2 points)**

Taking norms gives

$$\|(T - \lambda)^{-1} x\|_\infty = \sup_{n\in\mathbb{N}} \frac{|x_n|}{|\alpha^n - \lambda|} \le \frac{1}{\delta} \sup_{n\in\mathbb{N}} |x_n| = \frac{1}{\delta} \|x\|_\infty,$$

which shows that $(T - \lambda)^{-1}$ is bounded and thus $\lambda \in \rho(T)$.
**(2 points)**

Hence, we conclude that the spectrum of $T$ is given by $\sigma(T) = \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$.
**(5 points)**

**Solution of problem 3 (10 points)**

For any $x, y \in X$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}\|x+\lambda y\|^2 &= \langle x+\lambda y, x+\lambda y\rangle \\ &= \langle x, x+\lambda y\rangle + \langle \lambda y, x+\lambda y\rangle \\ &= \langle x, x\rangle + \langle x, \lambda y\rangle + \langle \lambda y, x\rangle + \langle \lambda y, \lambda y\rangle \\ &= \langle x, x\rangle + \overline{\lambda}\langle x, y\rangle + \lambda\langle y, x\rangle + |\lambda|^2\langle y, y\rangle.\end{aligned}$$

**(4 points)**

Multiplication with $\lambda$ gives

$$\lambda\|x+\lambda y\|^2 = \lambda\langle x, x\rangle + |\lambda|^2\langle x, y\rangle + \lambda^2\langle y, x\rangle + \lambda|\lambda|^2\langle y, y\rangle.$$

**(1 point)**

In particular, taking $\lambda \in \{1, i, -1, -i\}$ gives

$$\begin{aligned}\|x+y\|^2 &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle, \\ i\|x+iy\|^2 &= i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle + i\langle y, y\rangle, \\ -\|x-y\|^2 &= -\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle, \\ -i\|x-iy\|^2 &= -i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle - i\langle y, y\rangle.\end{aligned}$$

**(4 points)**

Adding the last four equalities gives the desired identity.
**(1 point)**

**Solution of problem 4 (5 + (5 + 5 + 5) = 25 points)**

(a) Let $X$ be a Banach space and let $Y$ be a normed linear space. Let $F \subset B(X, Y)$ and assume that

$$\sup_{T \in F} \|Tx\| < \infty \quad \text{for all} \quad x \in X.$$

Then the elements $T \in F$ are uniformly bounded:

$$\sup_{T \in F} \|T\| < \infty.$$

**(5 points)**

(b) (i) For all $p \in \mathcal{P}$ we have:

$$|T_n p| = \left| \sum_{k=0}^{n} a_k \right| \leq \sum_{k=0}^{n} |a_k| \leq (n+1) \max_{k=0,\ldots,n} |a_k| \leq (n+1)\|p\|.$$

We conclude that

$$\|T_n\| = \sup_{p \in \mathcal{P},\, p \neq 0} \frac{|T_n p|}{\|p\|} \leq n + 1.$$

**(4 points)**

On the other hand, for the polynomial $p(x) = 1 + x + \cdots + x^n$ we obviously have that $\|p\| = 1$ and

$$|T_n p_n| = \underbrace{1 + 1 + \cdots + 1}_{n+1 \text{ times}} = n + 1.$$

Hence, it follows that $\|T_n\| = n + 1$.
**(1 point)**

(ii) Take $C_p = \sum_{k=0}^{\infty} |a_k|$, where the $a_k$ are the coefficients of $p(x) = \sum_{k=0}^{\infty} a_k x^k$. Note that the infinite sum converges since only finitely many $a_k$ are nonzero. For every $n \in \mathbb{N}$ we have

$$|T_n p| = \left| \sum_{k=0}^{n} a_k \right| \leq \sum_{k=0}^{n} |a_k| \leq \sum_{k=0}^{\infty} |a_k| = C_p.$$

**(5 points)**

(iii) If the space $(\mathcal{P}, \|\cdot\|)$ *were* a Banach space, then an application of the Uniform Boundedness Principle with the set $F = \{T_n : n \in \mathbb{N}\} \subset B(\mathcal{P}, \mathbb{K})$ gives a contradiction.

Indeed, for every $p \in \mathcal{P}$ we have

$$\sup_{n \in \mathbb{N}} |T_n p| \leq \sup_{n \in \mathbb{N}} C_p = C_p < \infty.$$

**(2 points)**

So the Uniform Boundedness Principle would imply that

$$\sup_{n \in \mathbb{N}} \|T_n\| < \infty.$$

But this contradicts the fact that $\|T_n\| = n + 1$ for all $n \in \mathbb{N}$. Therefore, we conclude that $(\mathcal{P}, \|\cdot\|)$ is *not* a Banach space.
**(3 points)**

**Solution of problem 5 (5 + (8 + 7) = 20 points)**

(a) Let $X$ be a normed linear space and let $V \subset X$ be a linear subspace. If $f \in V'$, then there exists $F \in X'$ such that $F(v) = f(v)$ for all $v \in V$ and $\|F\| = \|f\|$.
**(5 points)**

(b) For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|f(\varphi)| = \left| \int_0^1 \varphi(t)\, dt \right| \leq \int_0^1 |\varphi(t)|\, dt \leq \int_0^1 \|\varphi\|_\infty \, dt = \|\varphi\|_\infty.$$

**(3 points)**

For the constant function $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|f(\varphi)| = 1$. Hence,

$$\|f\| = \sup_{\varphi \neq 0} \frac{|f(\varphi)|}{\|\varphi\|_\infty} = 1.$$

**(1 point)**

For $\varphi \in \mathcal{C}([0,1], \mathbb{K})$ we have that

$$|g(\varphi)| = |\varphi(c)| \leq \sup_{x \in [0,1]} |\varphi(x)| = \|\varphi\|_\infty.$$

**(3 points)**

For the constant function $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|g(\varphi)| = 1$. Hence,

$$\|g\| = \sup_{\varphi \neq 0} \frac{|g(\varphi)|}{\|\varphi\|_\infty} = 1.$$

**(1 point)**

(c) First observe that with $c = \frac{1}{2}$ it follows that $f(\varphi) = g(\varphi) = h(\varphi)$ for all $\varphi \in V$.
**(1 point)**

In particular, it then follows that

$$\|h\| = \sup_{\varphi \in V \setminus \{0\}} \frac{|h(\varphi)|}{\|\varphi\|_\infty} = \sup_{\varphi \in V \setminus \{0\}} \frac{|f(\varphi)|}{\|\varphi\|_\infty} \leq \sup_{\varphi \in \mathcal{C}([0,1],\mathbb{K}) \setminus \{0\}} \frac{|f(\varphi)|}{\|\varphi\|_\infty} = \|f\| = 1.$$

But note that with $\varphi(t) = 1$ we have $\|\varphi\|_\infty = 1$ and $|h(\varphi)| = 1$, which implies that $\|h\| = 1$.
**(4 points)**

We conclude that both $f$ and $g$ with $c = \frac{1}{2}$ are norm preserving extensions of $h$. But note that $f \neq g$, since for $\varphi(t) = t^2$ we have $f(\varphi) = \frac{1}{3}$ whereas $g(\varphi) = \frac{1}{4}$. Therefore, the norm preserving extension of $h$ obtained by the Hahn-Banach Theorem is not unique.
**(2 points)**